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Problems with the filling factor in the quantum Hall effect

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Abstract
Already in elementary quantum mechanics on the T^n inequivalent representations exist and one has to use reducible representations to implement some automorphisms unitarily. A special example is the quantum Hall system on the T^2. There exists a preferable representation for all B but in the passage to several particles the Pauli principle can be formulated in a physically satisfying way only if the commutant is abelian, i.e. for integer magnetic monopoles.

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1. Introduction

The equivalence of the two formulations of quantum mechanics, namely the Schrödinger equation and Heisenberg's commutation relations for finitely many particles seems to be guaranteed by von Neumann's theorem [1]. He showed that all irreducible representations of Heisenberg's commutation relations (in Weyl form) are unitarily equivalent to that proposed by Schrödinger. The theorem does not hold for infinitely many degrees of freedom and it is an essential feature of quantum field theory that different physical situations demand different representations. But also when the global structure of $\mathbb{R}^n$ is changed the theorem is not applicable any more. In this note we will study the problems one meets in elementary quantum mechanics on T^n, $n = 1, 2$.

(a) If the algebra of observables is not simple but has a centre there are many inequivalent irreducible representations, which assign different numerical values to the elements of the centre. A faithful representation is reducible. But also if the algebra has a trivial centre there are many inequivalent representations, irreducible and reducible ones, and both can be faithful.

(b) The time evolution will, in general, not be inner. If we want to represent it by a unitary operator we have to turn to a reducible representation.

The essential problem arises when we want to describe several fermions on T^n. Of course, in reality we work on $\mathbb{R}^n$. T^n only works as a finite approximation in order to get a well-defined thermodynamic limit when the size of the torus goes to infinity. Whether physics is described on T^n in a satisfying way can only be checked by examing whether it gives the correct thermodynamic limit in which the size of the system and the particle number tends to infinity. In order to describe several particles we take the tensor products of the representations and then apply the Pauli principle to the vectors. We apply this principle to various situations: On the T^1 this method only works in a physically satisfying way, if we deal with an irreducible representation. If we deal with a reducible representation we have to combine soldering of the centre with anti-symmetrization. We can also describe fermions on the torus by a modified CAR algebra (algebra of anti-commuting creation and annihilation operators).

On the T^2 we consider free particles in a constant magnetic field and construct the corresponding representations. For every strength of the magnetic field it is possible to construct irreducible one-particle representations. In these representations we can study the degeneracy of the Landau levels. This degeneracy is highly discontinuous for varying B. Only for an integer number of monopoles does the degeneracy converge in the thermodynamic limit to the correct particle density and can therefore be interpreted as a filling factor for the finite system. On the other hand, for every B there exist rather natural reducible representations that are continuous in the magnetic field. The special mathematical role of integer magnetic fields appears as the fact that the commutant (see section 2B and 3B) is abelian. The abelian commutant of several particles can be soldered, i.e. the commutants of different particles can be identified without spoiling the trivial commuation relations in the tensor product of the commutants. If the one-particle commutant is not abelian this is not possible anymore. Looking for a replacement of the soldering procedure, we can restrict the possible states over the n-fold tensor products by fixing the state over the commutants and applying the Pauli principle only to this restricted class of states. This gives a finite filling factor which is nevertheless discontinuous in B and therefore not in accordance with the physically desirable result. Thus, if we apply the methods that worked for T^1 to describe several fermions, we note that only for an integer number of magnetic monopoles [2] do we obtain the correct filling factor. Therefore, we are able to formulate the physics of a particle on a torus in a constant magnetic field in the language of C* algebras with time automorphism. But the definition of the corresponding second quantized CAR algebra works so far only for integer magnetic monopoles. How far a meaningful and satisfying deformation is possible for other B is the subject of further investigation.

2. Motion on T^1

2.1. Classical motion

The motion on a circle with a constant force E shows even at the classical level an unusual feature. The Hamiltonian

$$H_E = \frac{L^2}{2} - E\varphi \qquad \{\varphi, L\} = 1 \tag{2.1}$$

is not globally defined on T^1, but dH is and leads to the canonical flow Φ_E,

$$\begin{aligned} L(t) &= L(0) + E \cdot t \\ \varphi(t) &= \varphi(0) + tL(0) + E\frac{t^2}{2} = \varphi(0) + t\dot{\varphi}(0) + E\frac{t^2}{2}. \end{aligned} \tag{2.2}$$

There does not exist a constant of the motion, since H_E is only locally defined.

In physics a constant electric field on a circle can only be realized by a magnetic field increasing linearly in time. Thus another description of the situation is given by

$$H_M = \tfrac{1}{2}(L + Et)^2. \tag{2.3}$$

H_M is globally defined and leads to a time evolution

$$\begin{aligned} L(t) &= L(0) \\ \varphi(t) &= \varphi(0) + tL(0) + E\frac{t^2}{2} = \varphi(0) + t\dot{\varphi}(0) + E\frac{t^2}{2}. \end{aligned} \tag{2.4}$$

The flow $\Phi_M(t) : (\varphi, L) \to (\varphi(t), L(t))$ is a one-parameter family of canonical transformations, but not a group, $\Phi_M(t_1) \cdot \Phi_M(t_2) \neq \Phi_M(t_1 + t_2)$ and especially $\Phi_M(t) \circ \Phi_M(-t) \neq \mathbf{1}$. H_M is not even piece-wise constant.

H_E and H_M lead to the same motion of $\varphi(t)$ and they are gauge equivalent. A gauge flow Φ_G generated by

$$H_G = -E\varphi \qquad \Phi_G(t)(\varphi, L) = (\varphi, L + tE) \tag{2.5}$$

intertwines between them

$$\Phi_M(t) \circ \Phi_G(t) = \Phi_E(t). \tag{2.6}$$

$\Phi_G(t)$ does not commute with $\Phi_M(t)$ and restores the group property.

2.2. First quantization

The quantum mechanical observables of a particle on T^1 form the Weyl algebra $\mathcal{W}_T$ generated by

$$W(m, \alpha) = \mathrm{e}^{\mathrm{i}(m\varphi + \alpha L)} \qquad m \in \mathbb{Z} \quad \alpha \in \mathbb{R}. \tag{2.7}$$

It has the centre

$$\mathcal{Z} = \{W(0, \alpha); \alpha \in 2\pi\mathbb{Z}\}. \tag{2.8}$$

Correspondingly, there exists a one-parameter family of inequivalent irreducible representations Π_γ, $\gamma \in [0, 1)$, with

$$\Pi_\gamma(\mathrm{e}^{2\pi\mathrm{i}L}) = \mathrm{e}^{2\pi\mathrm{i}\gamma} \tag{2.9}$$

with γ the Bohm–Aharonov phase [3]. The spectrum of $\Pi_\gamma(L)$ is $\mathbb{Z} + \gamma$.

The flows Φ_E, Φ_M and Φ_G (2.2), (2.4) and (2.5) generate automorphisms on $\mathcal{W}_T$. However, Φ_E and Φ_G do not induce automorphisms of $\Pi\gamma(\mathcal{W}_T)$, since they do not leave the centre element-wise invariant. But in an irreducible representation the elements of the centre are represented by multiples of unity and cannot change under automorphisms. Only $\Phi_M(t) = \Phi_E(t) \circ \Phi_G(-t)$ induces an automorphism of W_T, but not a one-parameter group. For Φ_E to induce an automorphism one has to go to a faithful representation. This can be obtained by $\Pi = \int_0^{2\pi} d\gamma\, \Pi_\gamma$ (2.10) and it is clear that $\Phi_E(t)$ which moves in Π_γ to $\gamma + Et$ will induce an automorphism group. Π will be represented in a Hilbert space $\int_0^{2\pi} d\gamma \mathcal{H}_\gamma$ and this representation is equivalent to the standard representation Π_s of the Weyl algebra $\{W(m, \alpha), m, \alpha \in \mathbb{R}\}$ on $L^2(\mathbb{R})$ restricted to $\mathcal{W}_T$. Since Π_s is faithful so is its restriction to $\mathcal{W}_T$, but this restriction is no longer irreducible. To substantiate these claims we exhibit these restrictions in terms of wavefunctions.

2.3. Wave functions

It is known that $\Pi_\gamma(L)$ is self-adjoint on the domain $\mathcal{D}_\gamma$ in the Hilbert space $L^2(T^1) \equiv L^2([0, 2\pi), \mathrm{d}x)$

$$\mathcal{D}_\gamma = \{\psi \in L^2(T^1), \psi' \in L^2(T^1), \lim_{\varepsilon \to 0} \psi(2\pi - \varepsilon) = \mathrm{e}^{2\pi\mathrm{i}\gamma}\psi(0)\}. \tag{2.10}$$

If the vector $|n\rangle$ corresponds to the wavefunction $e^{i(n+\gamma)\varphi} \in \mathcal{D}_\gamma$, then we have the representation

$$\Pi_\gamma(e^{im\varphi})|n\rangle = |n+m\rangle \qquad \Pi_\gamma(e^{i\alpha L})|n\rangle = e^{i(n+\gamma)\alpha}|n\rangle. \tag{2.11}$$

Thus all Π_γ act in the same Hilbert space $L^2(T^1) = \ell^2$. Here eigenvectors of L belonging to different Π_γ are not orthogonal as in the reducible representation Π. To realize that we have to turn to a bigger space. $\Pi = \int_0^1 d\gamma\,\Pi_\gamma$ acts in the Hilbert space $\mathcal{H} = \ell^2 \otimes L^2(T^1)$ with elements $\sum |n\rangle \otimes \psi_n(2\pi\gamma)$, such that $e^{i\alpha L}$ is the multiplication operator $e^{i\alpha(n+\gamma)}$. This Hilbert space $\mathcal{H}$ can be identified with $L^2(\mathbb{R})$ by cutting $\mathbb{R}$ into pieces of length 1, $\mathbb{R} \ni p = [p]+(p)$, $[p] \leqslant p < [p+1]$, $[p] \in \mathbb{Z}$, $(p) =: \gamma \in [0, 1)$ and with

$$\tilde{\psi}(p) = \tilde{\psi}([p]+\gamma) \equiv |[p]\rangle \otimes \psi_{[p]}(2\pi\gamma) \equiv \psi([p], 2\pi\gamma). \tag{2.12}$$

This bijection is an isometry because

$$\int_{-\infty}^{+\infty} dp|\tilde{\psi}(p)|^2 = \sum_{[p]} \int_0^1 d\gamma|\tilde{\psi}([p]+\gamma)|^2.$$

The actions of $e^{in\varphi}$ and $e^{i\alpha L}$ correspond to the standard representations

$$\begin{aligned}
&e^{i\alpha L}\tilde{\psi}(p) = e^{i\alpha p}\tilde{\psi}(p) \Leftrightarrow e^{i\alpha L}\psi(n, 2\pi\gamma) = e^{i\alpha(n+\gamma)}\psi(n, 2\pi\gamma)\\
&e^{i\beta\varphi}\tilde{\psi}(p) = \tilde{\psi}(p+\beta) \Leftrightarrow e^{i\beta\varphi}\psi(n, 2\pi\gamma) = \psi([n+\gamma+\beta], (2\pi(\gamma+\beta))\\
&\qquad\qquad = \psi(n+[\gamma+\beta], 2\pi(\gamma+\beta))
\end{aligned} \tag{2.13}$$

which coincides for $\beta \in \mathbb{Z}$ with (2.11).

It remains to be shown that in the representation Π the automorphisms (2.2), (2.4) and (2.5) are unitarily implemented.

H_M generates the unitaries

$$U_M(t) = \exp\left[-i\int_0^t \frac{i}{2}(L+Et')^2\,dt'\right] = \exp\left[-i\frac{t}{2}(L^2+EtL) - iE\frac{t^3}{6}\right]. \tag{2.14}$$

The last factor as a pure c-number does not contribute to the automorphism and can be dropped.

$$U_G(t) = e^{iEt\varphi} \tag{2.15}$$

which according to (2.13) is well defined. Together they generate

$$U_t\tilde{\psi}(p) = U_G(t)U_M(t)\tilde{\psi}(p) = \exp\left[-i\frac{t}{2}(p^2+Etp)\right]\tilde{\psi}(p+Et). \tag{2.16}$$

In order to verify (2.6) one readily checks

$$\begin{aligned}
&e^{i\beta\varphi}U_t\tilde{\psi}(p) = U_t e^{i\beta(\varphi+pt+Et^2/2)}\tilde{\psi}(p)\\
&e^{i\alpha L}U_t\tilde{\psi}(p) = U_t e^{i\alpha(p+Et)}\tilde{\psi}(p).
\end{aligned} \tag{2.17}$$

The phase factor cancels out and U_t induces a one-parameter group of automorphisms, though $U_{t_1+t_2} = U_{t_1} \cdot U_{t_2}$ only modulo a phase factor.

Remark 1. In an irreducible representation Π_γ we identified L with a self-adjoint operator. Similarly we can identify φ with a bounded self-adjoint (multiplication operator) and interpret H_E in this sense. This operator $H_E = L^2/2 + E\varphi$ has a purely discrete spectrum, since according to the Golden–Thompson–Symanzik inequality

$$\mathrm{Tr}\, e^{-\beta(L^2/2+E\varphi)} \leqslant \mathrm{Tr}\, e^{-\beta L^2/2}\, e^{-\beta E\varphi} \leqslant e^{-\beta E\pi}\, \mathrm{Tr}\, e^{-\beta L^2/2} < \infty.$$

Thus the time evolution generated by $e^{it(L^2/2+E\varphi)}$ is quasi-periodic and has no resemblance of Φ_E to (2.2).

Remark 2. The reducible representation corresponds to the kq-representation of a particle in a periodic potential, periodic in $[0, 2\pi]$ in $L^2(\mathbb{R})$ [4]. Similarly, the Bohm–Aharonov phase as an effect of an electric field [5] is interpreted here as describing the flow through different irreducible representations.

Remark 3. Here we observe a different way of symmetry breaking.

Definitions

(i) A symmetry α is an automorphism of a C*-algebra $\mathcal{A}$.
(ii) A dynamical symmetry α of a dynamical system $(\mathcal{A}, \tau_t)$, where τ_t is the time evolution, is a symmetry with $\alpha \circ \tau = \tau \circ \alpha$.
(iii) The dynamical symmetry α is spontaneously broken by a state ω over $\mathcal{A}$, if $\omega = \omega \circ \tau \neq \omega \circ \alpha$.
(iv) The symmetry α is spontaneously destroyed by a representation Π if α is an automorphism of $\Pi(\mathcal{A})$ but cannot be extended to the weak closure $\Pi(\mathcal{A})''$.
(v) The symmetry α is completely destroyed by a representation Π, if it cannot be restricted to $\Pi(\mathcal{A})$.

Explanations

(iii) Since ω is time invariant τ is unitarily implemetable whereas α may or may not. For finite dimensional matrix algebras it certainly is and already then definition (iii) can be realized: Let $\vec{\sigma}_1, \vec{\sigma}_2$ be two sets of Pauli matrices, $\mathcal{A} = \{\mathbf{1}, \vec{\sigma}_1, \vec{\sigma}_2\}$. An example is provided by

$$\tau_t = \operatorname{ad}\exp[\mathrm{i}\vec{\sigma}_1\vec{\sigma}_2 t] \qquad \alpha = \operatorname{ad}\exp[\mathrm{i}\gamma(\vec{\sigma}_{1z} + \vec{\sigma}_{2z})]$$
$$\omega(a) = \operatorname{Tr}\frac{1+\sigma_{1x}}{2}\cdot\frac{1+\sigma_{2x}}{2}a.$$

(iv) Here we have the situation where α cannot be unitarily implemented because otherwise it could be extended to $\Pi(\mathcal{A})''$. For obvious reasons we call it 'anti-Wignerian'. This does not happen for matrix algebras where the weak closure does not go beyond norm closure but can happen in quantum field theory.
(v) This happens if $\mathcal{A}$ has a centre on which α acts nontrivially and Π is not faithful. It occurs for τ in the Weyl algebra over T^1 but also already in finite dimensions: Consider $\mathcal{A} = (\mathbf{1}, \vec{\sigma}, \tau_3)$, $\tau_3^2 = \mathbf{1}$. $\alpha(\mathbf{1}, \vec{\sigma}, \tau_3) = (\mathbf{1}, \vec{\sigma}, -\tau_3)$ cannot be realized in $\Pi(\mathbf{1}, \vec{\sigma}, \tau_3) = (\mathbf{1}, \vec{\sigma}, \mathbf{1})$.

2.4. Second Quantization

The standard procedure to do the second quantization on T^1 is to consider the CAR algebra over $\mathcal{L}^2(T^1)$. But this means that we already choose an irreducible representation Π_γ, and this choice determines the action of the Weyl operators as automorphisms on the $a(f)$.

For finitely many fermions N we can consider as a representation

$$\bigotimes_{i=1}^{n}\Pi_\gamma^{(i)}$$

for the symmetrized Weyl operators acting on the totally anti-symmetric $\bigwedge_{i=1}^{n}\mathcal{H}_\gamma^{(i)}$.

If we start with the reducible representation $\Pi = \int \mathrm{d}\gamma\,\Pi_\gamma$ on $\ell^2 \otimes L^2(T^1)$, then $(\ell^2 \otimes L^2(T^1)) \wedge (\ell^2 \otimes L^2(T^1))$ would allow both particles to be in the $L = 0$ state by anti-symmetrizing in $L^2(T^1) \otimes L^2(T^1)$. To prevent this we can work in

$$(\ell^2 \wedge \ell^2) \otimes L^2(T^1).$$

Pictorially we have to solder the two spaces together. In terms of physics soldering means that though changing in time the Bohm–Aharonov phase γ has to be the same for all particles. For many fermions this soldering is conveniently carried out in second quantization. This is accomplished by working with the algebra $\mathcal{A} = \mathrm{CAR}(\ell^2) \otimes \mathrm{C}(T^1)$ rather than $\mathrm{CAR}(\ell^2 \otimes L^2(T^1))$.

Therefore the anti-commutation relations read

$$\begin{aligned} &[a_n^* c(\gamma), a_m c'(\gamma)]_+ = c(\gamma)c'(\gamma)\delta_{nm} \\ &[a_n c(\gamma), a_m c'(\gamma)]_+ = 0 \quad n, m \in \mathbb{Z} \quad c(\gamma), c'(\gamma) \in C(T^1). \end{aligned} \tag{2.18}$$

The actions of the operators $\mathrm{e}^{\mathrm{i}\alpha L}$, $\mathrm{e}^{\mathrm{i}m\varphi}$ can be transferred to inner automorphisms of $\mathcal{A}$ by

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\alpha L} a_n c(\gamma) \mathrm{e}^{-\mathrm{i}\alpha L} = \mathrm{e}^{\mathrm{i}\alpha(n+\gamma)} a_n c(\gamma) \\ &\mathrm{e}^{\mathrm{i}m\varphi} a_n c(\gamma) \mathrm{e}^{-\mathrm{i}m\varphi} = a_{n+m} c(\gamma) \end{aligned} \tag{2.19}$$

whereas $\mathrm{e}^{\mathrm{i}\beta\varphi}$ defines in the same way

$$\mathrm{e}^{\mathrm{i}\beta\varphi} a_n c(\gamma) = a_{[n+\beta]}\, c(\beta + \gamma) \tag{2.20}$$

a gauge automorphism on the algebra, i.e. an automorphism that is not inner and gives rise to inequivalent representations (this is what gauge transformations do). Obviously the anti-commutation relations (2.18) are left invariant. Since the time evolution is composed of $\mathrm{e}^{\mathrm{i}\beta\varphi}$ and $\exp\left[\mathrm{i}\int_0^t \mathrm{d}t'(L + t'E)^2\,\mathrm{d}t'\right]$ it also induces an automorphism on the CAR algebra that is not inner. Note that $a_1^* c(\gamma) a_1^* g(\gamma) = a_1^{*2} c(\gamma) g(\gamma) = 0$ such that one cannot fill two electrons into the state $L = 1$.

Remark 4. That $\mathrm{e}^{\mathrm{i}\beta\varphi}$ cannot define an inner automorphism is due to the fact that $[n + \beta]$ is not continuous and there is no identification $a_n c(\gamma + 2\pi) = a_{n+1} c(\gamma)$. Thus $\mathrm{e}^{\mathrm{i}\beta\varphi}$ cannot be a quasifree automorphism $a_f \to a_{U_\beta f}$, f from some Hilbert space and U_β a strongly continuous unitarily group in it.

3. The quantum Hall effect

As a next example we study fermions on T^2. If they move freely, $T^2 = T^1 \otimes T^1$ and also $\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}H_1 t} \otimes \mathrm{e}^{\mathrm{i}H_1 t}$, therefore the analysis can be completely carried over. But if H does not factorize as happens in the quantum Hall effect, i.e. when we consider particles in a constant magnetic field new features will appear.

The quantum Hall effect is fairly well understood on the few-particle level [6] as well as on the level of quantum field theory [7] from the geometrical viewpoint. But the passage from one level to the other is not sufficiently under control. Of course, we have the description of many-particle wavefunctions [8] that explain some of the essential features. Also in [11] a thermodynamic limit is carried through, where the particles are confined by a harmonic potential that can be interpreted to be the result of a background charge and at the same time defines an electric field that induces a Hall current. But we would like to decouple the two facts and control the thermodynamic limit in other settings also. The usual approach is to confine the particles to some region in space, usually a box, whose volume increases proportional to the particle number. Then one has to choose boundary conditions for the Hamiltonian, in general Dirichlet, Neumann or periodic boundary conditions. These periodic boundary conditions are especially favourable. On the one hand, the calculation is usually easy, but in addition space translation still commutes with time evolution and therefore they are tailormade to allow currents. Thus we consider the motion on T^2 as being worth examining. As for particles on T^1

we will first describe the classical motion of particles in a constant magnetic field. Then we will do first quantization, i.e. define the one-particle Weyl algebra with its automorphisms and representation. Then we will consider the second quantization on $L(\mathbb{R}^2)$ and compare the possibilities of second quantization on $L^2(T^2)$.

3.1. Classical motion

As for the one-dimensional torus the relevant features already occur at the classical level, since all relevant Poisson brackets are c-numbers and therefore the first quantization is straightforward.

The canonical momenta $p_{1,2}$ have to be replaced by the gauge-invariant velocities $v_{1,2} = p_{1,2} \pm B\varphi_{2,1}/2$, which are canonically conjugate

$$\{v_1, v_2\} = B. \tag{3.1}$$

Another independent canonical pair is offered by the centre of the Larmor orbits (for $E = 0$)

$$\bar{\varphi}_{1,2} = \varphi_{1,2} \pm \frac{1}{B} v_{2,1} \qquad \{\bar{\varphi}_1, \bar{\varphi}_2\} = -\frac{1}{B} \qquad \{\bar{\varphi}_i, v_k\} = 0. \tag{3.2}$$

As in T^1 the Hamiltonian H_E with a nonvanishing electric field is defined only locally, whereas dH_E exists globally. We assume that E points in the 1-direction so that

$$H_E = \frac{1}{2}\left(v_1^2 + v_2^2\right) + E\left(\bar{\varphi}_1 - \frac{v_2}{B}\right) \qquad E > 0. \tag{3.3}$$

The corresponding equations of motion

$$\dot{v}_1 = Bv_2 + E \qquad \dot{v}_2 = -Bv_1 \qquad \dot{\bar{\varphi}}_1 = 0 \qquad \dot{\bar{\varphi}}_2 = \frac{E}{B} \tag{3.4}$$

generate the flow $\Phi_E(t)$

$$(\bar{\varphi}_1, \bar{\varphi}_2, v_1, v_2) \to (\bar{\varphi}_1, \bar{\varphi}_2 + tE/B, cv_1 + s(v_2 + E/B), c(v_2 + E/B) - s(v_1 - E/B)) \tag{3.5}$$

where $c = \cos Bt$, $s = \sin Bt$.

Taking into account that the constant electric field can be generated by a time-dependent vector potential $A = -Et$ an alternative description is furnished by

$$H_M = \tfrac{1}{2}\left((v_1 + Et)^2 + v_2^2\right) \tag{3.6}$$

which generates a flow $\Phi_M(t)$

$$(\bar{\varphi}_1, \bar{\varphi}_2, v_1, v_2) \to (\bar{\varphi}_1, \bar{\varphi}_2, cv_1 + s(v_2 + E/B) - Et,\ c(v_2 + E/B) - s(v_1 - E/B)). \tag{3.7}$$

We see that for this flow H_M is globally defined, but $\Phi_M(t)$ again does not form a group. Only combined with the gauge transformation $\Phi_G(\mathrm{t})$

$$(\bar{\varphi}_1, \bar{\varphi}_2, v_1, v_2) \to (\bar{\varphi}_1, \bar{\varphi}_2 + tE/B, v_1 + tE, v_2) \tag{3.8}$$

generated locally by

$$H_G = -\varphi_1 E = (-\bar{\varphi}_1 + v_2/B)E \tag{3.9}$$

we get the group $\Phi_E(t)$

$$\Phi_E = \Phi_G \circ \Phi_M. \tag{3.10}$$

3.2. First quantization

Though at the classical level the situation is very similar to the one-dimensional T^1 at the quantum level, different features appear for different values of B. The classical Poisson brackets are readily expressed as (multiplication) commutators in the Weyl form

$$\begin{aligned} e^{i\alpha v_1} e^{i\beta v_2} &= e^{-i\alpha\beta B} e^{i\beta v_2} e^{i\alpha v_1} \qquad && \alpha, \beta \in \mathbb{R} \\ e^{im\bar{\varphi}_1} e^{in\bar{\varphi}_2} &= e^{i(mn)/B} e^{in\bar{\varphi}_2} e^{im\bar{\varphi}_1} \qquad && m, n \in \mathbb{Z}. \end{aligned} \tag{3.11}$$

Depending on whether B is rational or irrational, we have different representations.

(a) If $B = (2\pi)^{-1} g_1/g_2$, $g_i \in \mathbb{Z}$, then the algebra has a centre (the operators in the algebra which commute with all others)

$$\mathcal{Z} = \{e^{im\bar{\varphi}_1 \cdot g_1}, e^{in\bar{\varphi}_2 \cdot g_1}\}. \tag{3.12}$$

In this case the algebra is equivalent to

$$M_{g_1 \times g_1} \otimes \mathcal{Z} \otimes \left\{e^{i\alpha v_1}, e^{i\beta v_2}\right\} \tag{3.13}$$

where $M_{g_1 \times g_1}$ is a full matrix algebra of dimension g_1 and possesses for the $\bar{\varphi}$-part g_1-dimensional irreducible matrix representations, e.g. (up to unitary equivalence)

$$\begin{aligned} (e^{i\bar{\varphi}_1})_{rs} &= \delta_{rs} \exp\left[i \frac{2\pi \cdot r g_2}{g_1}\right] \qquad && r, s = 1, \ldots, g_1 \\ (e^{i\bar{\varphi}_2})_{rs} &= \delta_{r,s+1} && r, s = 1, \ldots, g_1 \bmod g_2. \end{aligned} \tag{3.14}$$

If B is irrational the centre is trivial and $\{e^{im\bar{\varphi}_1}, e^{in\bar{\varphi}_2}\}$ allows irreducible representations which are faithful, namely e.g. on $L^2(T^1)$,

$$\Pi_a(e^{in\bar{\varphi}_2}) = \exp\left[-in \frac{1}{B} p_{\bar{\varphi}_1}\right] \qquad \Pi_a(e^{im\bar{\varphi}_1}) = e^{im\bar{\varphi}_1}. \tag{3.15}$$

$p_{\bar{\varphi}_1} = \frac{1}{i} \frac{\partial}{\partial \bar{\varphi}_1}$ with periodic boundary conditions. Now the commutant of $\{e^{im\bar{\varphi}_1}, e^{in\bar{\varphi}_2}\}$ is trivial and therefore this representation is irreducible. Performing the gauge automorphisms $\Theta_{\alpha,\beta}$, Θ_T:

$$\begin{aligned} \{e^{im\bar{\varphi}_1}, e^{in\bar{\varphi}_2}\} &\stackrel{\Theta_{\alpha,\beta}}{\to} \{e^{im(\bar{\varphi}_1+\alpha)}, e^{in(\bar{\varphi}_2+\beta)}\} \\ \{e^{im\bar{\varphi}_1}, e^{in\bar{\varphi}_2}\} &\stackrel{\Theta_T}{\to} \left\{\exp\left[i\left(\left\langle\begin{smallmatrix}1\\0\end{smallmatrix}\right|T\left|\begin{smallmatrix}m\\n\end{smallmatrix}\right\rangle \bar{\varphi}_1 + \left\langle\begin{smallmatrix}0\\1\end{smallmatrix}\right|T\left|\begin{smallmatrix}m\\n\end{smallmatrix}\right\rangle \bar{\varphi}_2\right)\right]\right\} \end{aligned} \tag{3.16}$$

with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1 \qquad \left|\begin{smallmatrix}m\\n\end{smallmatrix}\right\rangle \in \mathbb{Z}^2$$

we obtain inequivalent irreducible representations. Integrating over them we obtain a faithful reducible representation Π.

These irreducible representations can also be described in the appendix in the language of wavefunctions.

Therefore in the sense of the possible ways of symmetry breaking, the gauge transformation $\bar{\varphi}_{1,2} \to \bar{\varphi}_{1,2} + \alpha_{1,2}$ is completely destroyed if B is rational and spontaneously destroyed if B is irrational because in the latter case $\Pi_a(e^{in\bar{\varphi}_2})$ changes its spectrum and thus the transformation is not unitarily implementable.

(b) There also exists a reducible representation on $L^2(T^2)$, namely

$$\begin{aligned}\Pi_b(\mathrm{e}^{\mathrm{i}\bar\varphi_1}) &= \exp\left[\mathrm{i}\left(x + \frac{1}{2B}\ p_y\right)\right]\\ \Pi_b(\mathrm{e}^{\mathrm{i}\bar\varphi_2}) &= \exp\left[\mathrm{i}\left(y - \frac{1}{2B}\ p_x\right)\right].\end{aligned} \tag{3.17}$$

This algebra $\Pi_b(\mathcal{A})''$ is type II_1. The tracial state is furnished by the identity function $|1\rangle$ on T^2

$$\langle 1|\mathrm{e}^{\mathrm{i}n\bar\varphi_1}\mathrm{e}^{\mathrm{i}m\bar\varphi_2}|1\rangle = \delta_{n0}\delta_{m0}. \tag{3.18}$$

and its commutant $\Pi_b(\mathcal{A})'$ (i.e. the operators in $\mathcal{L}^2(T^2)$ that commute with all operators in $\Pi_b(\mathcal{A})''$) is given by

$$\Pi_b(\mathcal{A})' = \left\{\exp\left[\mathrm{i}\left(x - \frac{1}{2B}\ p_y\right)\right], \exp\left[\mathrm{i}\left(y + \frac{1}{2B}\ p_x\right)\right]\right\}. \tag{3.19}$$

Note that this representation also works, if $2\pi B$ is rational. In this case Π_b is reducible and contains a centre $\mathcal{Z} = \Pi_b(\mathcal{A})'' \cap \Pi_b(\mathcal{A})'$

$$\Pi_b(\mathcal{A})'' \cap \Pi_b(\mathcal{A})' = \left\{\exp\left[\mathrm{i}g_1\left(x - \frac{1}{2B}p_y\right)\right], \exp\left[\mathrm{i}g_1\left(y + \frac{1}{2B}p_x\right)\right]\right\} = \mathcal{Z} \neq c\,\mathbf{1}. \tag{3.20}$$

Varying over B the representation Π_b is strongly continuous in B for $1/B \in (0, 1)$ and periodic in $2\pi/B$. Since we have in mind to perform the thermodynamic limit where B has to be replaced by BL^2, we approach the limit of continuity.

(c) There exists yet another representation [9] where continuity is preserved, namely on $L^2(\mathbb{R})$ where $\mathcal{A}$ is considered as a subalgebra of the Weyl algebra on $\mathbb{R}$ [9]

$$\begin{aligned}\Pi_c(\mathrm{e}^{\mathrm{i}n\bar\varphi_1}) &= \mathrm{e}^{\mathrm{i}xn/\sqrt{B}}\\ \Pi_c(\mathrm{e}^{\mathrm{i}m\bar\varphi_2}) &= \mathrm{e}^{-\mathrm{i}mp_x/\sqrt{B}}.\end{aligned} \tag{3.21}$$

The commutant $\Pi_c(\mathcal{A})'$ is given by

$$\left\{\mathrm{e}^{2\pi\mathrm{i}nx\sqrt{B}}, \mathrm{e}^{2\pi\mathrm{i}mp_x\sqrt{B}}\right\}''. \tag{3.22}$$

The elements of both $\Pi_c(\mathcal{A})$ and $\Pi_c(\mathcal{A})''$ are strongly continuous in $1/B \in (0, \infty)$ and in the thermodynamic limit $B \to BL^2, L \to \infty$, we have

$$\begin{aligned}&\text{st-}\lim_{B\to\infty} \mathrm{e}^{\mathrm{i}xn/\sqrt{B}}\mathrm{e}^{\mathrm{i}mp_x/\sqrt{B}} = 1 \quad \text{for the operators in } \Pi_c(\mathcal{A})\\ &\text{w-}\lim_{B\to\infty} \mathrm{e}^{2\pi\mathrm{i}n\sqrt{B}x}\mathrm{e}^{2\pi\mathrm{i}m\sqrt{B}p_x} = \delta_{n0}\delta_{m0} \quad \text{for the operators in } \Pi_c(\mathcal{A})'\\ &\text{st-}\lim \exp\left[\mathrm{i}x\frac{[n\sqrt{B}]}{\sqrt{B}}\right] = \mathrm{e}^{\mathrm{i}nx}\\ &\text{st-}\lim \exp\left[\mathrm{i}p_x\frac{[m\sqrt{B}]}{\sqrt{B}}\right] = \mathrm{e}^{\mathrm{i}mp_x}.\end{aligned} \tag{3.23}$$

The representation has the advantage that the correct filling factor $B \cdot 2\pi (= \rho(x)(2\pi)^2$ in (3.30)) appears as a characteristic of the representation, namely as the coupling constant [10].

It is defined as follows: take a vector ψ from the representation space $\mathcal{H}$ of $\mathcal{M} = \Pi(\mathcal{A})''$ and define the projections $P'_\psi \in \mathcal{M}'$ (resp. $P_\psi \in \mathcal{M}$) by $\overline{\mathcal{M}\psi} = P'_\psi\mathcal{H}$

(resp. $\overline{\mathcal{M}'\psi} = P_\psi \mathcal{H}$). Furthermore, assume that there are normalized traces τ (resp. τ') over $\mathcal{M}$ (resp. $\mathcal{M}'$) with $\tau|_{\mathcal{Z}} = \tau'|_{\mathcal{Z}}$. Then von Neumann found that

$$c = \tau(P_\psi)/\tau'(P'_\psi) \tag{3.24}$$

is independent of ψ and thus is a characteristic of the relation between $\mathcal{M}$ and $\mathcal{M}'$. Furthermore $\exists\ \psi =$ cyclic for $\mathcal{M} \Rightarrow c \leqslant 1$, $\exists\ \psi =$ separating for $\mathcal{M} \Rightarrow c \geqslant 1$.

Examples for c (3.25)

1. Π_a: $\mathcal{M} = M_{g_1\times g_1} \otimes \mathcal{Z}$, $\mathcal{M}' = \mathbf{1} \otimes \mathcal{Z}$. Here the traces on $\mathcal{M}$ and $\mathcal{M}'$ are unique up to a state on $\mathcal{Z}$ that has to coincide. Take $\psi = e_i \otimes v$, e_i any vector of C^{g1}, v a cyclic vector for $\mathcal{Z}$. Then $P'_\psi = \mathbf{1}$, $P_\psi = |e_i\rangle\langle e_i| \otimes \mathbf{1}$ and $c = \tau(P_\psi)/\tau'(P'_\psi) = 1/g_1$. Thus for $2\pi B$ integer $= g_1$ the filling factor c^{-1} gives the desired result, for $2\pi B = g_1/g_2$, $g_2 \neq 1$, it is too big.
2. Π_{b}: Here the vector $\psi = |1\rangle$ is cyclic and separating for $\mathcal{M}$ and $\mathcal{M}'$, i.e. $\overline{\mathcal{M}\psi} = \overline{\mathcal{M}'\psi} = \mathcal{H} \Rightarrow P_\psi = P'_\psi = \mathbf{1} \Rightarrow c = 1$.
3. Π_{c}: Take $2\pi B \geqslant 1$ (for $2\pi B \leqslant 1$ $\mathcal{M} \leftrightarrow \mathcal{M}'$) and for ψ the characteristic function $\chi_{[0,1/\sqrt{B}]} \in L^2(\mathbb{R})$. Then

$$\left\langle \psi | \mathrm{e}^{2\pi \mathrm{i} n x\sqrt{B}} \mathrm{e}^{2\pi \mathrm{i} p_x \sqrt{B}} |\psi \right\rangle = \delta_{n,0}\delta_{m,0} \tag{3.26}$$

and ψ implements the tracial state over $\mathcal{M}'$. Therefore, it is separating for $\mathcal{M}'$ and $P'_\psi = \mathbf{1}$. For $\mathcal{M}$ ψ is cyclic but not separating and the projection operator

$$P_\psi = \chi_{[0,1/\sqrt{B}]\cup[2\pi\sqrt{B},2\pi\sqrt{B}+1/\sqrt{B}]\cup\ldots} \in \mathcal{M} \tag{3.27}$$

satisfies $P_\psi\mathcal{H} = \mathcal{M}'|\psi\rangle$. If $\mathcal{M}'$ is type II_1 algebra (i.e. $2\pi B$ irrational), so is $\mathcal{M} = \mathcal{M}''$ and there exists a unique trace over $\mathcal{M}$. The trace is shift invariant and when applied to P_ψ gives the ratio of the intervals where $P_\psi = \mathbf{1}$ to the gaps between them. Thus

$$c = \tau(P_\psi) = \frac{1}{2\pi B}. \tag{3.28}$$

If $B/2\pi = g_1/g_2$ is rational, then

$$\mathcal{M} = M_{g_1\times g_1} \otimes \mathcal{Z} \qquad \mathcal{M}' = M_{g_2\times g_2} \otimes \mathcal{Z},$$

and again, with $\mathcal{Z} = \{\mathrm{e}^{\mathrm{i}n\sqrt{g_1}\,x}\}''$

$$c = g_2/g_1.$$

The automorphisms corresponding to Φ_M (3.6) are inner and therefore unitarily implementable in all three representations Π_a, Π_b, Π_c.

The automorphisms Φ_G and Φ_E are not inner and therefore are not unitarily implementable in Π_a (compare (3.16)). The representations Π_b and Π_c are the tracial representation resp. quasi-equivalent (i.e. the state $\langle 1|\Pi_b(\mathrm{e}^{\mathrm{i}n\bar\varphi_1 + m\bar\varphi_2})|1\rangle$ can be obtained by a density operator over $\Pi_c(\mathcal{M})''$) to the tracial representation. There the automorphisms Φ_G and Φ_E can be extended to the weak closure and therefore are unitarily implementable in Π_b, resp. Π_c.

3.3. Second quantization

One of the desiderata of the second quantization is that for fermions we get the correct filling factor for the Landau levels. In the thermodynamic limit when $T^2 \to \mathbb{R}^2$ the representation of the Weyl algebra is essentially unique and the second quantization works in the standard way.

The ground states of the fermions for the time evolution Φ_M with $E = 0$ is given by the quasi-free state

$$\omega(a(f)a^\dagger(g)) = \langle f|P_{v_x,v_y} \otimes A_{\bar{x},\bar{y}}|g\rangle \tag{3.29}$$

where P_{v_x,v_y} is the projection on the ground state in the $\mathcal{H}(v_x, v_y)$ space whereas $A_{\bar{x},\bar{y}}$ is some positive operator $\leqslant 1$ in the $\mathcal{H}(\bar{x}, \bar{y})$ space. The maximal filling factor is obtained for $A = \mathbf{1}$. If we write $P \otimes A$ as an integral kernel in the $\mathcal{H}(x, y)$ space, $K(x, y)$, then $K(x, x) = \rho(x)$ gives the particle density. With

$$\mathbf{1}_{\bar{x}} = \lim_{E\to\infty} \left(\Theta\left(E - \mathrm{e}^{-(\bar{x}^2+\bar{y}^2)}\right)\right.$$

this integral kernel can be evaluated [11] to be

$$K(x, x) = \frac{B}{2\pi}. \tag{3.30}$$

If we now turn to the second quantization on the torus, we have to distinguish between different cases.

(a) $B = g_1/2\pi$, $g_1 \in \mathbb{Z}$.
Here we can apply the methods of T^1. Either we introduce creation and annihilation operators over the representation space $\mathcal{H}_a$ and accept that the gauge automorphisms are not defined on the CAR algebra. Or we can choose the representation Π_b. In this case the algebra contains a centre, but the commutant is not abelian. The centre can be soldered but it is not clear how to solder the remaining part of the commutant.
In the representation Π_c the commutant is again abelian and we can solder it in analogy to (2.18). Now the one-particle Hilbert space can be written as $\mathcal{H} = L^2(\mathbb{R}) \otimes sC^{g_1} \otimes L^2(\mathcal{Z})$ and the creation and annihilation operators satisfy with $f \in \mathcal{L}^2(\mathbb{R})$, $v \in C^{g_1}$, $c(z) \in \mathcal{L}^\infty(\mathcal{Z})$,

$$[a(f \otimes v)c(z), a^\dagger(f' \otimes v')c'(z)]_+ = \langle f|f'\rangle\langle v|v'\rangle \cdot c(z)c'(z). \tag{3.31}$$

If the vacuum $|0\rangle$ is defined by

$$a(\psi)|0\rangle = 0$$

then

$$\prod_{k\in I} a^\dagger(f \otimes v_k)c_k(z)|0\rangle = 0 \tag{3.32}$$

if $|I| > g_1$. Therefore the filling density $g_1/(2\pi)^2 = B/2\pi$. (We take into account that our torus has length 2π.)
The time evolution Φ_M extends to an automorphism over this algebra and has the form

$$\tau_t a(f \otimes v_k)c(\gamma_1, \gamma_2) = a(U_t f \otimes \mathrm{e}^{\mathrm{i}tE/B} v_k)c(\gamma_1, \gamma_2 + tE/2\pi). \tag{3.33}$$

(b) Take $B = 2\pi g_1/g_2$, $g_1, g_2 \in \mathcal{Z}$, $g_2 \neq 1$, $g_2 < g_1$.
An irreducible representation Π_a acts on a Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes C^{g_1}$. If we do a second quantization over this Hilbert space, then we obtain creation operators $a^\dagger(f \otimes v)$, $v \in C^{g_1}$ and get a filling density

$$\frac{g_1}{(2\pi)^2} = \frac{Bg_2}{2\pi} \tag{3.34}$$

which can become arbitrarily big, independent of the strength B. This suggests that we have to start with a representation of the one-particle algebra, where the filling factor already arises as a natural parameter, namely Π_c.

(c) We take arbitrary B and start with the representation Π_c. In the Hilbert space $\mathcal{H}_c$ we can find vectors $\psi_1, \ldots \psi_k$, $k \leqslant g_1/g_2$, such that ψ_i implements the tracial state on $\mathcal{M}'$ and satisfy $P_{\psi_i} \cdot P_{\psi_j} = \delta_{ij} P_{\psi_i}$, e.g. we can choose $\psi_i = \chi_{[(i-1)/\sqrt{B}, i/\sqrt{B}]}$. Then we consider the vector $\bar{\psi}$ in the Hilbert space $\bar{\mathcal{H}} = (\bigotimes L^2(\mathbb{R})^n)$

$$\bar{\psi} = \sum (-1)^\pi \ \psi_{\pi(1)} \otimes \ldots \psi_{\pi(n)}. \tag{3.35}$$

This procedure only works, if $n \leqslant g_1/g_2$. In addition, it has the advantage that the set of permitted states $\bar{\psi}$ is mapped into itself under unitary operators that implement one-particle automorphisms. Especially for the time evolution Φ_M with $E = 0$ we can find a sequence $\bar{\psi}_L$ in $L^2\left(T_L^{(B/L^2)}\right)$ that defines the thermodynamic limit state.

But the restriction on the set $\bar{\psi}$ has also a severe shortcoming: it is not stable under linear superposition nor is it stable under unitary operators of $\bigotimes_s \{\Pi_c(W)\}^{(n)}$. Therefore ground states with respect to Hamiltonians that include particle interactions will, in general, not belong to this class of vectors. The set of states that is stable under unitary operators of $\mathcal{M}_n''$ is characterized by its action on $\mathcal{M}_n' = (\otimes \mathcal{M}_1' \ldots \otimes \mathcal{M}_n') \vee \{U_\pi\}''$, i.e.

$$\langle \bar{\psi} | M_1' \otimes \ldots M_n' | \bar{\psi} \rangle = \prod \langle \psi_i | M_i' | \psi_i \rangle = \prod \tau(M_i'). \tag{3.36}$$

We check if we can find vectors satisfying (3.36) in $L^2\left(T_L^N\right)$, even if $N \gg B/L^2$. For simplicity we assume that $2\pi B$ is rational and we have already soldered the centre. Therefore we start with a one-particle Hilbert space $\mathcal{H}^{g_1 \cdot g_2}$ with $\mathcal{B}(\mathcal{H}) = \mathcal{M} \otimes \mathcal{M}'$, $\mathcal{M} \approx M_{g_1 \times g_1}$, $\mathcal{M}' \approx M_{g_2 \times g_2}$. For integer L we consider the sequence

$$\mathcal{M} \approx M_{g_1 L^2 \times g_1 L^2} \qquad \mathcal{M}' \approx M_{g_2 \times g_2}. \tag{3.37}$$

For $\mathcal{H}^{g_2}$ we choose a basis $\bar{e}_1, \ldots \bar{e}_{g_2}$. Then $\bar{\psi}$ is of the form

$$\bar{\psi} = \left| \sum_{i_1 \ldots i_N} \sum e_{i_1} \otimes \ldots e_{i_N} \otimes \bar{e}_{j_1} \otimes \ldots \bar{e}_{j_n} \right\rangle = \left| \sum_I e_I \otimes \bar{e}_I \right\rangle \tag{3.38}$$

with $1 \leqslant j_i \leqslant g_2$ and the restriction $\langle e_I | e_{I'} \rangle = \delta_{II'}$.

The Pauli principle has to be expressed as a restriction on the permitted class of e_I. We demand that e_I has to be anti-symmetric in $\mathcal{H}^{g_1 \cdot L^2}$. Therefore

$$\binom{g_1 \cdot L^2}{N} \geqslant g_2^N. \tag{3.39}$$

If we assume that N scales like $k L^2$, then (3.39) becomes in the limit $L \to \infty$

$$g_1 \ln g_1 - k \ln k - (g_1 - k) \ln(g_1 - k) \geqslant k \ln g_2. \tag{3.40}$$

For $g_2 = 1$, $g_1 = 2\pi B$ this implements $k \leqslant 2\pi B$, the correct filling factor. But, in general, k will depend on both g_1 and g_2 and in the limit $g_2 \to \infty$ (B becoming irrational)

$$k_{\text{crit}} = \frac{g_1}{g_2} \cdot e.$$

Interestingly enough the condition (3.36) implies that the filling factor remains finite, i.e. this method is closer to reality than doing a second quantization on irreducible representations but it still does not give the correct result (3.34). This means that either we find a better way than (3.36) of soldering the commutant, when it is not abelian, or we accept that on the torus the magnetic monopoles that produce the magnetic field have to be quantized.

Appendix

We shall now exhibit the vectors of the representation space by functions $f(\varphi_1, \varphi_2) \in L^2(T^2)$. There it is customary to use a nonsymmetrical gauge

$$v_1 = p_1 + B\varphi_2 \qquad v_2 = p_2 \qquad \bar{\varphi}_1 = \varphi_1 + \frac{p_2}{B} \qquad \bar{\varphi}_2 = -\frac{p_1}{B} \qquad p_i = -\mathrm{i}\frac{\partial}{\partial \varphi_i}. \tag{A.1}$$

The action of the Weyl operators in $L^2(T^2)$ are

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\alpha v_1} f(\varphi_1, \varphi_2) &= \mathrm{e}^{\mathrm{i}\alpha B\varphi_2} f(\varphi_1 + \alpha, \varphi_2) \\ \mathrm{e}^{\mathrm{i}\beta v_2} f(\varphi_1, \varphi_2) &= f(\varphi_1, \varphi_2 + \beta) \\ \mathrm{e}^{\mathrm{i}m\bar{\varphi}_2} f(\varphi_1, \varphi_2) &= f(\varphi_1 - m/B, \varphi_2) \\ \mathrm{e}^{\mathrm{i}n\bar{\varphi}_1} f(\varphi_1, \varphi_2) &= \mathrm{e}^{\mathrm{i}n\varphi_1} f(\varphi_1, \varphi_2 + n/B). \end{aligned} \tag{A.2}$$

Here $\alpha, \beta \in \mathbb{R}$, $m, n \in \mathbb{Z}$, so the infinitesimal generators (A.1) for $\bar{\varphi}_1$ need not exist but for v_j they have to. We are interested in the lowest level of $\frac{1}{2}\left(v_1^2 + v_2^2\right)$ which corresponds to f's satisfying

$$(v_1 - \mathrm{i}v_2) f = 0. \tag{A.3}$$

To solve this equation it is convenient to introduce the half-sided Fourier series

$$f(\varphi_1, \varphi_2) = \sum_{k \in \mathbf{Z}} \mathrm{e}^{\mathrm{i}k\varphi_1} b_k(\varphi_2) \tag{A.4}$$

which changes (A.3) into

$$(k + B\varphi_2) b_k(\varphi_2) = \frac{\partial}{\partial \varphi_2} b_k(\varphi_2) \tag{A.5}$$

which is solved by

$$b_k(\varphi_2) = \mathrm{e}^{-\frac{1}{2}(k + B\varphi_2)^2} c_k. \tag{A.6}$$

We still have to make sure that these functions are in the domain of self-adjointness of the v_j (or equivalently of the p_j). Since $-\mathrm{i}\,\partial/\partial\varphi$ is self-adjoint for $f(2\pi) = \mathrm{e}^{\mathrm{i}\gamma} f(0)$ there is no problem with v_1 since (A.4) implies $f(0, \varphi_2) = f(2\pi, \varphi_2)$. For v_2 we have to require $f(\varphi, 2\pi) = \mathrm{e}^{\mathrm{i}\gamma(\varphi_1)} f(\varphi_1, 0)$ or for (A.6)

$$\sum_k c_k \mathrm{e}^{\mathrm{i}k\varphi_1} \{\mathrm{e}^{-k^2/2} \mathrm{e}^{\mathrm{i}\gamma(\varphi_1)} - \mathrm{e}^{-(k + 2\pi B)^2/2}\} = 0. \tag{A.7}$$

Now we specialize to $B = g_1/2\pi g_2$, $g_i \in \mathbf{Z}^+$ and see how functions of the form (A.6) can give the representation Π_a. Exchanging $(1 \leftrightarrow 2)$ we have the requirements

(a) $\mathrm{e}^{\mathrm{i}\bar{\varphi}_2} f_j = \mathrm{e}^{2\pi \mathrm{i} j/g_1} f_j$, $j = 1, 2, \ldots g_1$.
For (A.6) this means

$$\sum_k c_k \mathrm{e}^{\mathrm{i}k(\varphi, -1/B)} \mathrm{e}^{-(k + \beta\varphi_2)^2/2} = \sum_k c_k \mathrm{e}^{\mathrm{i}(k\varphi_1 + 2\pi j/g_1)} \mathrm{e}^{-(k + B\varphi_2)^2/2}.$$

Thus we have to restrict ourselves to those k for which there exists n $\in \mathbb{Z}$ such that $-2\pi k g_2/g_1 = 2\pi j/g_1 + 2\pi n$ or to the set $S_n(j) = \{k \in \mathbb{Z} : \exists\, n \in \mathbb{Z}$ with $-kg_2 = j + ng_1\}$. Note that $\forall\, k \in \mathbb{Z}$ there exists exactly one pair (j, n) such that this relation holds. For us j is fixed so we have to add the smaller set $S_n(j)$ to n.

(b) $e^{-i\tilde{\varphi}_1} f_j = f_{j+1} (\bmod g_1)$.

$$\sum_{k\in S_n(j)} c_k e^{i\varphi_1(k-1)} e^{-(k+B(\varphi_2-1/B))^2/2} = \sum_{k\in S_n(j+1)} c_k e^{i\varphi_1 k} e^{-(k+B\varphi_2)^2/2}.$$

The left side equals

$$\sum_{k+1\in S_n(j)} c_{k+1} e^{i\varphi_1 k}\ e^{-(k+B\varphi_2)^2/2}$$

so equality holds if c_k is independent of k and

$$\{k : -(k+1)g_2 = j + ng_1\} = \{k : -kg_2 = j + 1 + mg_1\}$$

or $g_2 = 1 + g\ g_1$ for some $g \in \mathbb{Z}$.

We still have to see for which g_2 (A.7) can be satisfied. It requires

$$\sum_{k\in S_n(j)} e^{ik\varphi_1} \left\{ e^{-k^2/2}\ e^{i\gamma(\varphi_1)} - e^{-(k+g_1/g_2)^2/2} \right\} = 0.$$

For $g_2 = 1$ this condition can be satisfied by $e^{i\gamma(\varphi_1)} = e^{ig_1\varphi_1}$, since $k \in S_n(j) \Rightarrow k - g_1 \in S_n(j)$. For $g_2 = 1 + g\, g_1$, $g \neq 0$, there is no chance that this holds for all φ_1. In particular for $\varphi_1 = 0$ all terms are positive and even for

$$f(x) = \sum_{k\in\mathbb{Z}} e^{-(k+x)^2/2} < f(0) \qquad \forall\, x \in (0, 1).$$

This calculation gives an explicit verification of the claim made in [2], namely that the Schrödinger equation on T^2 requires $B = \mathcal{Z}/2\pi$. This does not mean that for $g_2 \neq 1$ the operators (A.2) cannot be represented in some $L^2(T^2)$:

If we move to the representation Π_a for $B = 2\pi g_1/g_2$, $g_2 \neq 1$, we can represent it in $L^2(T^2)g_2$. Introducing $\varphi_2 g_2 = \tilde{\varphi}_2$ we take the same representation on $[0, 2\pi] \times [0, 2\pi g_2]$ with the necessary scaling, i.e.

$$e^{i\alpha v} f(\varphi_1, \tilde{\varphi}_2) = e^{i\alpha g_1 \tilde{\varphi}_2} f(\varphi_1 + \alpha, \tilde{\varphi}_2) = e^{i\alpha B\varphi_2} f(\varphi_1 + \alpha, \tilde{\varphi}_2).$$

Comparing Π_a for $b = 2\pi g_1$ with that for $B = 2\pi g_1/g_2$

$$\Pi_{a,B=2\pi g_1}(e^{in\varphi_1}, e^{im\varphi_2}) \text{ is non-degenerate,}$$

whereas

$$\Pi_{a,B=2\pi g_1/g_2}(e^{in\varphi_1}, e^{im\varphi_2}) \text{ is } g_2 \text{ times degenerate.}$$

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